Funciones complejas de variable real.

$w:[a,b] \to \mathbb{C}$,

$w(t) = u(t) + i\,v(t)$

Derivada: $\lim_{t \to t_0} \frac{w(t)-w(t_0)}{t-t_0} = w'(t_0) = u'(t_0) + i\,v'(t_0)$

Primitiva de $w(t)$ es $W(t)$ tal que $W'(t) = w(t)$

Ejemplo: $w(t) = e^{i\alpha t} = \cos(\alpha t) + i\,\mathrm{sen}(\alpha t)$

$w'(t) = -\alpha\,\mathrm{sen}(\alpha t) + i\,\alpha\cos(\alpha t) = \alpha i\,(\cos(\alpha t) + i\,\mathrm{sen}(\alpha t))$

$w'(t) = i\alpha e^{i\alpha t}$

Primitiva: $W(t) = \frac{e^{i\alpha t}}{i\alpha}$ ya que $W'(t) = \frac{1}{i\alpha}\left(i\alpha e^{i\alpha t}\right) = w(t)$

Ejemplo: $w(t) = (1-t^2) + i\,2t$

$w'(t) = -2t + 2i$

Primitiva: $W(t) = t - \frac{t^3}{3} + i\,t^2$, ya que

$W'(t) = 1 - t^2 + i\,2t = w(t)$

Integral de función compleja de variable real.

$w:[a,b] \to \mathbb{C}$, $w(t) = u(t) + i v(t)$

$$\int_a^b w(t)\,dt = \underbrace{\int_a^b u(t)\,dt}_{\mathrm{Re}\int_a^b w(t)dt} + i \underbrace{\int_a^b v(t)\,dt}_{\mathrm{Im}\int_a^b w(t)dt}$$

Existencia? $\to$ si w es continua a trozos ($\Leftrightarrow$ u y v continuas a trozos)

Ejemplo: $\omega(t) = 1-t^2 + i\,2t$ $0 \le t \le 1$

$$\int_0^1 \omega(t)\,dt = \int_0^1 1-t^2\,dt + i\int_0^1 2t\,dt = t - \frac{t^3}{3}\Big|_0^1 + i\,t^2\Big|_0^1 =$$

$$= 1 - \frac{1}{3} + i = \frac{2}{3} + i$$

Propiedades

- $\int_a^b z_0\,\omega(t)\,dt = z_0\int_a^b \omega(t)\,dt$
- $\int_a^b \omega(t) + z(t)\,dt = \int_a^b \omega(t)\,dt + \int_a^b z(t)\,dt$
- $\int_a^b \omega(t)\,dt = \int_a^c \omega(t)\,dt + \int_c^b \omega(t)\,dt$
- Si $W(t)$ es primitiva de $\omega(t)$: $\int_a^b \omega(t)\,dt = W(b) - W(a)$
- $\left|\int_a^b \omega(t)\,dt\right| \le \int_a^b |\omega(t)|\,dt$

Ejemplos $\int_0^\pi e^{2it}\,dt = \frac{e^{2it}}{2i}\Big|_0^\pi = \frac{e^{i2\pi} - e^0}{2i} = 0$

$$\int_0^1 (1+it)^2\,dt = \frac{(1+it)^3}{3i}\Big|_0^1 = \frac{(1+i)^3 - 1}{3i} = \frac{1+3i-3-i-1}{3i} =$$

$$= \frac{3i-3-i}{3i} = \frac{2}{3} + i$$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Obs: $(1+it)^2 = 1-t^2 + i\,2t$ (ver el ejemplo al inicio de esta pág.)

$$\int_0^\pi e^x\cos x\,dx + i\int_0^\pi e^x \operatorname{sen} x\,dx = \int_0^\pi e^{(1+i)x}\,dx = \frac{e^{(1+i)x}}{1+i}\Big|_0^\pi =$$

$$= \frac{e^{(1+i)\pi} - 1}{1+i} = \frac{e^\pi(\cos\pi + i\operatorname{sen}\pi) - 1}{1+i} = \frac{-(e^\pi+1)(1-i)}{2} = \frac{-e^\pi - 1}{2} + i\,\frac{(1+e^\pi)}{2}$$

Resulta:

$$\int_0^{\pi} e^x \cos x \, dx = \frac{-e^{\pi}-1}{2}$$

$$\int_0^{\pi} e^x \operatorname{sen} x \, dx = \frac{1+e^{\pi}}{2}$$

Curvas en el plano complejo

$x = x(t)$
$y = y(t)$ $\quad$ $x(t), y(t)$ continuas en $[a,b]$.

$Z = z(t) = x(t) + i\,y(t)$ $\rightarrow$ describe una curva en plano

Curva simple: si no se corta a sí misma:

$z(t_1) \neq z(t_2)$ si $t_1 \neq t_2$
$(t_1 \neq b, t_2 \neq b)$

Curva cerrada simple: curva simple, con $z(a) = z(b)$

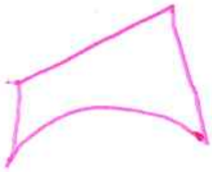
Curva de Jordan

Teorema de la curva de Jordan: Toda curva cerrada simple en el plano divide a éste en dos abiertos disjuntos: uno acotado, llamado recinto interior (RI(C)) y uno no acotado, llamado recinto exterior (RE(C))

Curva regular: si $z(t)$ es C^1 en (a,b) y $z'(t) \neq 0$

Curva regular por tramos: si es C^1 en (a,b), excepto en una cantidad finita de puntos, en los que z' tiene discontinuidad de salto o evitable; y $z'(t) \neq 0$, excepto en una cantidad finita de puntos

Curva regular



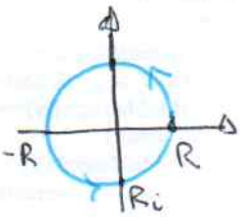
curva regular por tramos, simple, cerrada

Curva regular por tramos : CONTORNO

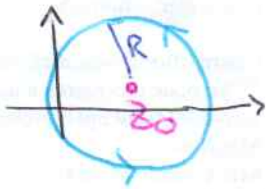
Ejemplos

(1) $z(t) = R\cos t + i R \operatorname{sen} t \quad t \in [0, 2\pi]$

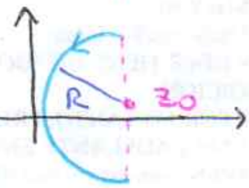
$z(t) = Re^{it}$



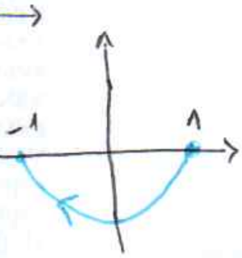
(2) $z(t) = z_0 + Re^{it} \quad t \in [0, 2\pi]$



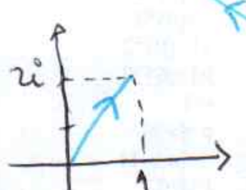
(3) $z(t) = z_0 + Re^{it} \quad t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$



(4) $z(t) = e^{-2it} \quad t \in [0, \pi/2]$



(5) $z(t) = (1+2i)t \quad t \in [0,1]$



Repaso: integral de línea de campos vectoriales.

C: curva parametrizada: $\gamma(t) = (x(t), y(t))$ $t \in [a,b]$

Longitud: $\boxed{L = \int_C dl = \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{x'^2(t) + y'^2(t)}\, dt}$

$\bar{F}: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $\bar{F}(x,y) = (P(x,y), Q(x,y))$

$$\boxed{\int_C \bar{F}\cdot d\bar{s} = \int_a^b \bar{F}(\gamma(t))\cdot \gamma'(t)\, dt = \int_a^b (P(\gamma(t))\cdot x'(t) + Q(\gamma(t))\cdot y'(t))dt}$$

$\int_C \bar{F}\cdot d\bar{s} = \int_C P\,dx + Q\,dy$ → notación

↳ circulación

Propiedades

lineal: $\int_C (\bar{F} + \bar{G})\cdot d\bar{s} = \int_C \bar{F}\, d\bar{s} + \int_C \bar{G}\, d\bar{s}$

aditiva: $C = C_1 + C_2 \Rightarrow \int_C \bar{F}\cdot d\bar{s} = \int_{C_1} \bar{F}\cdot d\bar{s} + \int_{C_2} \bar{F}\cdot d\bar{s}$

$\int_C \bar{F}\cdot d\bar{s} = -\int_{-C} \bar{F}\cdot d\bar{s}$

independiente de la parametrización (si conserva orient)

Integrales de funciones complejas sobre contornos

Sea C contorno, parametrizado con $z(t)=x(t)+iy(t)$, $t\in[a,b]$.
f continua a trozos sobre C

$$\int_C f(z)\,dz = \int_a^b f(z(t)).z'(t)\,dt$$

Si $f(z)=u(x,y)+iv(x,y)$,
$z'(t)=x'(t)+iy'(t)$

$$\int_C f(z)\,dz = \int_a^b \big(u(x(t),y(t))+iv(x(t),y(t))\big).(x'(t)+iy'(t))\,dt$$

$$=\int_a^b u.x'-v.y'\,dt + i\int_a^b u.y'+v.x'\,dt$$

$$=\int_a^b (u,-v).(x',y')\,dt + i\int_a^b (v,u)(x',y')\,dt$$

$$=\underbrace{\int_C u\,dx+(-v)\,dy}_{\text{circulación de }(u,-v)} + i\underbrace{\int_C v\,dx+u\,dy}_{\text{circulación de }(v,u)}$$

Propiedades: las de integ. de campos vectoriales.

Mód: $\left|\int_C f(z)\,dz\right| = \left|\int_a^b \underbrace{f(z(t)).z'(t)}_{w(t)}\,dt\right| \leq \int_a^b \underbrace{|f(z(t))|.|z'(t)|}_{|w(t)|}\,dt$

Si $|f(z)|\leq M \quad \forall z\in C$:

$$\boxed{\left|\int_C f(z)\,dz\right| \leq \int_a^b M.|z'(t)|\,dt = M\underbrace{\int_a^b |z'(t)|\,dt}_{\text{long. } C = L} = M.L}$$

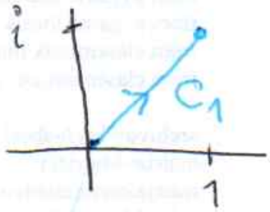
Ejemplos

① $\int_{C_1} \bar{z}\, dz$ con $C_1 :\ z(t) = (1+i)t\ ,\ t \in [0,1]$

$$\int_{C_1} \bar{z}\, dz = \int_0^1 (1-i)t\,(1+i)\, dt = 2\int_0^1 t\, dt = 2\cdot\frac{1}{2} = 1$$

$f(z) = \bar{z}\ ,\ f(z(t)) = \overline{(1+i)t}$

$z'(t) = 1+i$

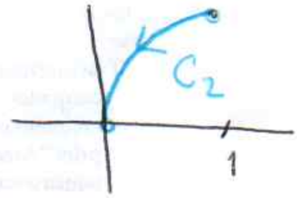


② $\int_{C_2} \bar{z}\, dz$ $\quad C_2 :\ z(t) = 1 + e^{it}\ ,\ t \in [\pi/2, \pi]$

$$\int_{C_2} \bar{z}\, dz = \int_{\pi/2}^{\pi} \overline{(1+e^{it})}\cdot \underbrace{ie^{it}}_{z'(t)}\, dt = \int_{\pi/2}^{\pi} (1+e^{-it})\, i e^{it}\, dt =$$

$$= i\int_{\pi/2}^{\pi} e^{it} + 1\, dt = i\left[\frac{e^{it}}{i} + t\right]_{\pi/2}^{\pi} = e^{i\pi} - e^{i\pi/2} + i\frac{\pi}{2} =$$

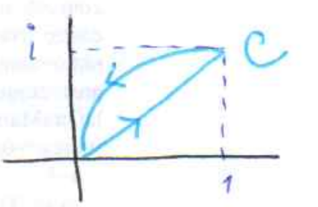
$$= -1 - i + i\frac{\pi}{2} = -1 + i\left(\frac{\pi}{2} - 1\right)$$



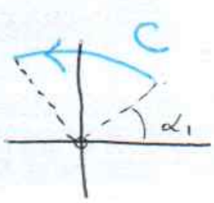
③ $\int_C \bar{z}\, dz$ $\quad C : C_1 + C_2$

$$\int_C \bar{z}\, dz = \int_{C_1} \bar{z}\, dz + \int_{C_2} \bar{z}\, dz = 1 + \left(-1 + i\left(\frac{\pi}{2} - 1\right)\right)$$

$$= i\left(\frac{\pi}{2} - 1\right)$$



④ $\int_C \frac{1}{z}\, dz$ $\quad C :\ z(t) = e^{it}\quad t \in [\alpha_1, \alpha_2]$



$$\int_C \frac{1}{z}\, dz = \int_{\alpha_1}^{\alpha_2} \frac{1}{e^{it}}\cdot i e^{it}\, dt = i\int_{\alpha_1}^{\alpha_2} dt = i(\alpha_2 - \alpha_1)$$

C : circunferencia positivamente orientada: $\boxed{\int_C \frac{1}{z}\, dz = 2\pi i}$

(5) $\int_C \frac{1}{z^2}\,dz \qquad C:\ z(t) = Re^{it} \qquad t \in [\alpha_1, \alpha_2]$

$$\int_C \frac{1}{z^2}\,dz = \int_{\alpha_1}^{\alpha_2} \frac{1}{R^2 e^{2it}} \underbrace{R i e^{it}}_{z'(t)}\,dt = \frac{i}{R}\int_{\alpha_1}^{\alpha_2} e^{-it}\,dt =$$

$$= \frac{i}{R}\,\frac{e^{-it}}{-i}\Big|_{\alpha_1}^{\alpha_2} = -\left(\frac{1}{Re^{i\alpha_2}} - \frac{1}{Re^{i\alpha_1}}\right) = -\left(\frac{1}{z}\Big|_{z(\alpha_2)} - \frac{1}{z}\Big|_{z(\alpha_1)}\right)$$

$$= -\frac{1}{z}\Big|_{z(\alpha_1)}^{z(\alpha_2)}$$

C: circunferencia positiva : $\int_C \frac{1}{z^2}\,dz = 0$

(6)

$\int_C 3z^2\,dz \qquad C:\ z(t),\ t \in [a,b].$

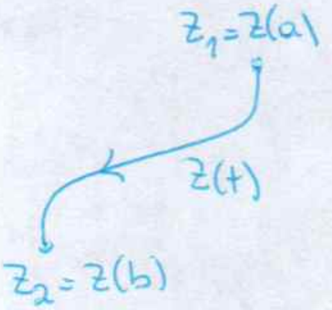
$$\int_C 3z^2\,dz = \int_a^b \underbrace{3z^2(t)\cdot z'(t)}_{w(t)}\,dt = \int_a^b W'(t)\,dt = W(b) - W(a) =$$

$$W(t) = z^3(t) \Rightarrow W'(t) = 3z^2(t)\cdot z'(t)$$

$$= z^3(b) - z^3(a) = z_2^3 - z_1^3 = F(z_2) - F(z_1)$$

pto inicial: $z_1 = z(a)$
pto final: $z_2 = z(b)$

$F(z) = z^3$, primitiva de $3z^2$



Recordemos (Análisis II)

Teorema de independencia del camino.

Sea $\bar{F}: D \subset \mathbb{R}^2 \to \mathbb{R}^2$ campo continuo en abierto conexo D.

Equivalen:

a) existe un campo escalar φ C^1 en D tal que
$$\nabla \varphi = \bar{F}$$

b) $\int_C \bar{F} \cdot d\bar{s}$ depende sólo del pto inicial y final de C
(no depende del recorrido de C)

c) $\int_C \bar{F} \cdot d\bar{s} = 0$ si C es contorno cerrado.

Además si ocurre a) (o b o c, ya que son equivalentes)

$\Rightarrow \underbrace{D\bar{F}}_{\text{matriz jacobiana.}}$ es simétrica

Si D es simplemente conexo y $D\bar{F}$ es simétrico en D

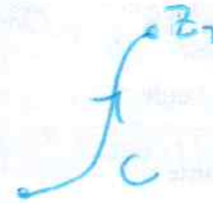
$\Rightarrow$ se cumple a) (y b) y c), ya que son equivalentes)

Sea $f: D \subset \mathbb{C} \to \mathbb{C}$, continua, D abierto conexo.

Si f tiene primitiva en D, $\left.\begin{array}{l} F'(z) = f(z) = u + iv \\ F(z) = U + iV \end{array}\right\} \Rightarrow \begin{array}{l} U'_x = u \\ V'_x = v \end{array}$

$$\int_C f(z)\,dz = \int_C (u, -v) \cdot d\bar{s} + i \int_C (v, u) \cdot d\bar{s} =$$

$$= \int_C (U'_x, -V'_x) \cdot d\bar{s} + i \int_C (V'_x, U'_x)\, d\bar{s} =$$



$$= \int_C (U'_x, U'_y) \cdot d\bar{s} + i \int_C (V'_x, V'_y) \cdot d\bar{s} =$$

$$= \int_C \nabla U \cdot d\bar{s} + i \int_C \nabla V \cdot d\bar{s}$$ $\Rightarrow$ no depende de recorrido, es nula si C es cerrada.

$$= U(z_2) - U(z_1) + i\left(V(z_2) - V(z_1)\right) = F(z_2) - F(z_1)$$

f tiene primitiva F en D

$\iff$

la integral sobre contorno C no depende de ~~contorno~~ recorrido, sólo de pto inicial y final

$\iff$ la integral sobre contorno cerrado es nula

y en este caso: $\displaystyle\int_C f(z)\,dz = \underbrace{F(z_2)}_{\text{pto fin}} - \underbrace{F(z_1)}_{\text{pto ini}}$

Ejemplos

① $\displaystyle\int_C \frac{1}{z^2}\,dz = -\frac{1}{z}\Big|_{\text{pto fin}} - \left(-\frac{1}{z}\right)\Big|_{\text{pto ini}}$

② $\displaystyle\int_C \operatorname{sen}(z)\,dz = -\cos z\Big|_{z(1)} - (-\cos z)\Big|_{z(0)} = -\cos(2) + \cos(i)$

$C:\ z(t) = 2t^2 + i(1-t)$

$t \in [0,1]$

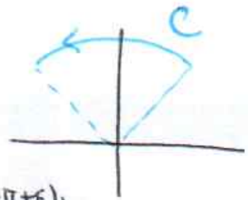
③ $\displaystyle\int_C \frac{1}{z}\,dz = \operatorname{Log}(z)\Big|_{z(\pi/4)}^{z(\frac{3\pi}{4})} = \operatorname{Log}\left(Re^{i\frac{3\pi}{4}}\right) - \operatorname{Log}\left(Re^{i\frac{\pi}{4}}\right)$

$C:\ z(t) = Re^{it}$

$t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$= \ln R + i\frac{3\pi}{4} - \left(\ln R + i\frac{\pi}{4}\right)$

$= i\frac{\pi}{2}$



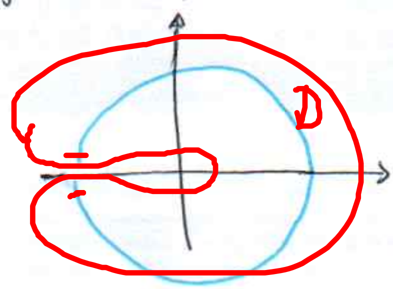
④ $\displaystyle\int_C \frac{1}{z}\,dz = \operatorname{Log}(z)\Big|_{z(-\pi+\varepsilon)}^{z(\pi-\varepsilon)} = \operatorname{Log}\left(Re^{i(\pi-\varepsilon)}\right) - \operatorname{Log}\left(Re^{i(-\pi+\varepsilon)}\right)$

$C:\ z(t) = Re^{it}$

$t \in (-\pi+\varepsilon, \pi-\varepsilon)$

$= i(\pi-\varepsilon) - i(-\pi+\varepsilon)$

$= 2\pi i - i2\varepsilon \xrightarrow[\varepsilon\to 0]{} 2\pi i$



$\displaystyle\lim_{\varepsilon\to 0} \boxed{\int_C \frac{1}{z}\,dz = 2\pi i}$